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Tunnelling Effect of Massive Particles from Kerr Black Holes

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In this paper, we extend Parikh's (massless particles) and Zhang' work to massive particles' Kerr black hole tunnelling. By treating the massive particle as de Broglie wave, we calculate the emission rates of the particles across the event horizon of the Kerr black holes. Our result is successful and is in agreement with the form of the massless particles.

KEY WORDS:

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1. INTRODUCTION

Recently, Parikh and Wilczek proposed a method in order to solve the information loss paradox (Parikh, 2004a,b; Parikh and Wilczek, 2000). They took the self-interaction effect into account and viewed Hawking radiation as a tunnelling process across the event horizon. In this way, they gave a leading correction to the semiclassical emission rate. Following this method, the tunnelling effect from some kinds of black holes has been investigated (Hemming and Keski-Vakkuri, 2001; Medved, 2002).

In 2005, the concerned research work got much progress. Zhang extended Parikh's work to the stationary axisymmetric Kerr black holes (Zhang and Zhao, 2005a) and massive particles' black hole tunnelling–de Sitter tunnelling (Zhang and Zhao, 2005b). However, in their work, the black hole is stationary axisymmetric (the particle is massless) or the particle is massive (the black holes are spherically symmetric). In this paper, we investigate the tunnelling effect not only from the stationary axisymmetric kerr black holes but also of massive particles. There are three crucial points. First, not only the total energy but also the total an-

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gular momentum must be conserved because the relation between the total angular momentum J and the total energy m is $J = ma$, where a is the angular momentum per unit mass of the black hole. That is to say, the total mass of the black hole will change when a particle tunnels out of the event horizon. And at the same time, the total angular momentum of the hole must change too. Second, in order to describe the process across the event horizon, we should adopt the Painlevè coordinates in which none of the components of the metric are singular. Due to Landau's condition of the coordinate clock synchronization, we could work out the phase and the group velocity. Third, for the sake of the simplicity, the massive particle is treated as the de Broglie wave. Then we calculate the emission rates of the massive particles which tunnel out of the event horizon by using Parikh' approach ((Parikh, 2004a)) and the result is successful. Throughout the paper, the units ($G = C = \hbar = 1$) are used.

2. PAINLEVÈ COORDINATES

The line elements of the Kerr black hole are

$$ds^2 = -\left(1 - \frac{2mr}{\rho^2}\right) dt_s^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left[(r^2 + a^2)\sin^2\theta + \frac{2mra^2\sin^4\theta}{\rho^2}\right] d\varphi^2 - \frac{4mra\sin^2\theta}{\rho^2} dt_s d\varphi, \quad (1)$$

where

$$\rho \equiv r^2 + a^2\cos^2\theta, \quad (2)$$

$$\Delta \equiv r^2 + a^2 - 2mr. \quad (3)$$

and m and a are the total mass and the angular momentum per unit mass of the black hole respectively. It is manifest that there is a coordinate singularity in the line elements (1) at the event horizon. However, we should choice a coordinate system which is well-behaved at the event horizon to study the process across the event horizon. We must adopt the dragging coordinate system first in order to obtain the Painlevè coordinates.

Let

$$\frac{d\varphi}{dt_s} = -\frac{g_{03}}{g_{33}}. \quad (4)$$

The line elements in dragging coordinates could be written as

$$ds^2 = -\frac{\rho^2\Delta}{(r^2 + a^2)^2 - \Delta a^2\sin^2\theta} dt_s^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2. \quad (5)$$

Then we introduce the new coordinate

$$t = t_s - \int [F(r, \theta)dr + G(r, \theta)\, d\theta]. \tag{6}$$

to get the Painlevè coordinates. The function $F(r, \theta)$ and $G(r, \theta)$ should satisfy $\frac{\partial F(r,\theta)}{\partial \theta} = \frac{\partial G(r,\theta)}{\partial r}$ because of the integrability condition (Zhang and Zhao, 2005a). After calculation, we could obtain the line element in new coordinates–Painlevè coordinates (Zhang and Zhao, 2005a)

$$\begin{aligned} ds^2 = &-\frac{\rho^2 \Delta}{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta} dt^2 + 2\sqrt{\frac{\rho^2(\rho^2 - \Delta)}{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}} dt dr + dr^2 \\ &+ g'_{22} d\theta^2 + 2g'_{12} dr d\theta + 2g'_{02} dt d\theta, \end{aligned} \tag{7}$$

In fact, we do not need to integrate Eq. (6) and do not know the exact form of g'_{22}, $2g'_{12}$ and $2g'_{02}$. It is obvious that there is no singularity in line element (7).

Because in quantum mechanics particles tunnelling through the barrier is an instantaneous process, the line elements (7) should satisfy the Landau’ condition of the coordinate clock synchronization (Landau and Lifshitz, 1975). Fortunately, the line elements (7) do satisfy Landau’ condition (Zhang and Zhao, 2005a). So we have found a coordinate which is suitable to study the tunnelling effect.

3. GEODESICS EQUATION OF THE MASSIVE PARTICLES

Because the geodesics of massive particles is not light-like, we could not obtain the radial geodesics equation by making $ds^2 = 0$ in Eq. (7). For the sake of simplicity, the outgoing particle is treated as de Broglie wave. It is well known that the wave equation could be written as approximately (Zhang and Zhao, 2005b)

$$\psi(r, t) = c \exp\left[i \left(\int_{r-\varepsilon}^{r} P_r dr - \omega t\right)\right] \tag{8}$$

where $r - \varepsilon$ represents the initial location of the particle. From

$$\int_{r-\varepsilon}^{r} P_r dr - \omega t = \phi_0 \tag{9}$$

we could get $\dot{r} \equiv \frac{dr}{dt} = \frac{\omega}{k}$, where k is the de Broglie wave number. It is obvious that $\dot{r}$ is the phase velocity in fact. It is well known that the relation between phase velocity v_p and the group velocity v_g of the de Broglie wave is

$$v_p = \frac{1}{2} v_g \tag{10}$$

$$v_p = \frac{dr}{dt} = \frac{\omega}{k}, \quad v_g = \frac{dr_c}{dt} = \frac{d\omega}{dk} \tag{11}$$

There are two events during the particles tunnel across the barrier. According to Landau' theory of the coordinate clock synchronization, the difference of coordinate times between the particle tunnelling into the barrier and the particle tunnelling out the barrier is

$$dt = -\frac{g_{0i}}{g_{00}}dx^i = -\frac{g_{01}}{g_{00}}dr_c \tag{12}$$

where $d\theta = 0$ in Eq. (7) and r_c is the location of the particle. Substituting the line elements (7) into Eq. (12) and (11), we can get the phase velocity

$$\dot{r} = v_p = \frac{1}{2}\frac{\rho^2\Delta}{\sqrt{\rho^2\left(\rho^2 - \Delta\right)}\sqrt{\left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2\theta}} \tag{13}$$

If the self-gravitation is taken into account, Eqs. (7) and (13) should be replaced by $m \to m - \omega$, where ω is the energy of the massive particle.

4. TUNNELLING RATE

In terms of WKB approximation and the Lagrangian theory, the imaginary part of the action could be written as (Parikh, 2004a; Zhang and Zhao, 2005a)

$$\begin{aligned} ImS &= Im\left[\int_{r_{\rm in}}^{r_{\rm out}} P_r dr + \int_{\varphi_{\rm in}}^{\varphi_{\rm out}} (-P_\varphi) d\varphi\right] \\ &= Im\left[\int_{r_{\rm in}}^{r_{\rm out}} \int_0^{P_r} dP_r' dr - \int_{\varphi_{\rm in}}^{\varphi_{\rm out}} \int_0^{P_\varphi} dP_\varphi' d\varphi\right], \end{aligned} \tag{14}$$

where P_r and P_φ are two canonical momentum conjugates to r and φ respectively. When the Hamilton's Equations, $\dot{r} = \frac{dH}{dP_r}\mid_{(r;\varphi,P_\varphi)}$ and $\dot{\varphi} = \frac{dH}{dP_\varphi}\mid_{(\varphi;r,P_r)} = \frac{\Omega_H' dJ}{dP_\varphi} = a\Omega_H'\frac{d(m-\omega')}{dP_\varphi} = -a\Omega_H'\frac{d\omega'}{dP_\varphi}$, are used, we could get

$$ImS = Im\left[\int_0^{\omega}\int_{r_{\rm in}}^{r_{\rm out}} \frac{dr}{\dot{r}} d(-\omega') - \int_0^{\omega}\int_{\varphi_{\rm in}}^{\varphi_{\rm out}} \frac{d\varphi}{\dot{\varphi}} a\Omega_H' d(-\omega')\right]. \tag{15}$$

where $dH_{(\varphi;r,P_r)} = \Omega_H' dJ = a\Omega_H' d(m - \omega')$ represents energy change of hole because of the loss of the angular momentum (Bardeen *et al.*, 1973) when a particle tunnels out of the black hole. From $\dot{\varphi} = \frac{d\varphi}{dt}$ and $\dot{r} = \frac{dr}{dt}$, we could obtain $\frac{d\varphi}{\dot{\varphi}} = \frac{dr}{\dot{r}}$. So Eq. (15) could be rewritten as

$$ImS = Im\left[\int_0^{\omega}\int_{r_{\rm in}}^{r_{\rm out}} \frac{dr}{\dot{r}} d(-\omega') - \int_0^{\omega}\int_{r_{\rm in}}^{r_{\rm out}} \frac{dr}{\dot{r}} a\Omega_H' d(-\omega')\right] \tag{16}$$

where $\Omega_H' = \frac{a}{(r_+')^2 + a^2} = \frac{a}{2(m-\omega')^2 + 2(m-\omega')\sqrt{(m-\omega')^2 - a^2}}$ is the dragging angular velocity of the Kerr black hole. Substituting line elements (7) into Eq. (16), we could obtain

$$ImS = Im\left\{\int_0^{\omega}\int_{r_{in}}^{r_{out}} \frac{2drd(m-\omega')\sqrt{\rho^2(\rho^2-\Delta')}\sqrt{\left(r^2+a^2\right)^2-\Delta' a^2\sin^2\theta}}{\rho^2\Delta'}\left[1-\frac{a^2}{(r'_+)^2+a^2}\right]\right\} \tag{17}$$

where

$$\Delta' = r^2 + a^2 - 2(m-\omega')r = (r - r'_+)(r - r'_-), \tag{18}$$

$$r_{\text{in}} = m + \sqrt{m^2 - a^2}, \tag{19}$$

$$r_{\text{out}} = m - \omega + \sqrt{(m-\omega)^2 - a^2}, \tag{20}$$

It is obvious that $r = r'_+ = m - \omega' + \sqrt{(m-\omega')^2 - a^2}$ is a pole. The integral could be evaluated by deforming the contour around the pole, in this way we obtain

$$ImS = \int_m^{m-\omega} -\frac{2\pi[(m-\omega')^2 + (m-\omega')\sqrt{(m-\omega')^2-a^2}]}{\sqrt{(m-\omega')^2-a^2}}d(m-\omega') + \int_m^{m-\omega}\frac{\pi a^2}{\sqrt{(m-\omega')^2-a^2}}d(m-\omega')]. \tag{21}$$

At last, ImS could be written as

$$ImS = \pi[m^2 - (m-\omega)^2 + m\sqrt{m^2-a^2} - (m-\omega)\sqrt{(m-\omega)^2-a^2}]. \tag{22}$$

The entropy of the Kerr black hole before and after the emission are respectively

$$S_{\text{BH}_1} = \frac{1}{4}A_{H_1} = 2\pi[m^2 + m\sqrt{m^2-a^2}], \tag{23}$$

$$S_{\text{BH}_2} = \frac{1}{4}A_{H_2} = 2\pi[(m-\omega)^2 + (m-\omega)\sqrt{(m-\omega)^2-a^2}]. \tag{24}$$

It is manifest that the tunnelling rate is

$$\Gamma \sim \exp[-2ImS] = e^{\Delta S_{\text{BH}}}. \tag{25}$$

5. CONCLUSION

We have studied the tunnelling effect of massive particles from the Kerr black hole. In the calculation, the reason why we should consider the conservation of the total angular momentum is that there is a relation, $J = ma$, between the total angular momentum and the total energy. So, the total angular momentum must change corresponding to the change of the total energy. In addition, our results are in agreement with the underlying unitary theory.

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